The renormalized volume of hyperbolic 3-manifolds

Jean-Marc Schlenker

University of Luxembourg

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What is the renormalized volume?

General outline

In this section we will *not* define the renormalized volume, but rather explain its key properties (and why they are relevant).

- Originally defined by physicists (Hennigson-Skenderis). Motivation: action for gravity. More general context of conformally compact Einstein manifolds (= convex co-compact hyperbolic manifolds in dim 3).
- Brought into mathematics by Witten, Graham-Witten.
- Related to Liouville theory considered by complex analysists (Takhtajan, Teo, Zograf). Relation by Krasnov.
- Relations to 3d hyperbolic geometry, volume of the convex core, etc.
- Applications to 3-dimensional hyperbolic geometry (Bridgeman, Brock, Bromberg, Vargas-Pallete, etc).
- Recent applications back to physics (S.-Witten 2022).

Need to introduce basic definitions.

Complex structures on surfaces

We consider a closed, oriented surface S of genus at least 2.

Definition

A complex structure on S is an atlas with charts in $\mathbb{C}$ and transition maps which are biholomorphisms.

Definition

An *almost complex* structure on S is the choice of a bundle morphism $J : TS \to TS$ such that $J^2 = -I$.

Any complex structure determines an almost-complex structure, through multiplication by i. For surfaces (but not in higher dimensions) any almost-complex structure is "integrable", that is, comes from a complex structure.

Teichmüller space

Definition

The Teichmüller space of S, $\mathcal{T}_S$, is the quotient of the space of complex structures on S by the group of diffeomorphisms isotopic to the identity.

The group of diffeomorphisms of S, $\mathrm{Diff}(S)$, acts on the space of complex structures by pull-back.
The identity component Diff_0 of Diff is the group of diffeomorphisms isotopic to the identity.
Note that $\mathrm{Diff}(S)/\mathrm{Diff}_0(S)$ is the mapping-class group of S.

Complex structures and conformal structures

There is another equivalent point of view.

Definition

Two Riemannian metrics g, g' on S are *conformally equivalent* if there exists a function $u : S \to \mathbb{R}$ such that $g' = e^{2u} g$.

Definition

A *conformal class* (or *conformal structure* on S is an equivalence class of metrics, under the equivalence relation of conformality.

- Any almost-complex structure on S determines a conformal class on S: all Riemannian metrics g on S such that for all $(x, v) \in TS$, $g(Jv, Jv) = g(v, v)$.
- If S is *orientable*, any conformal structure c on S determines a unique almost-complex structure J: rotation of angle $\pi/2$ for g, where g is any metric in the conformal class c.

The hyperbolic plane

We need another point of view on Teichmüller space, as the space of hyperbolic metrics on S.

Definition

The hyperbolic plane $\mathbb{H}^2$ is the unique complete, simply connected surface of constant curvature -1.

Different models:

- As a quadric in the 3-dimensional Minkowski space $\mathbb{R}^{2,1}$, with the induced metric: $\{x \in \mathbb{R}^{2,1} \mid \langle x, x \rangle = -1 \ \wedge x_0 > 0\}$.
- In the *Poincaré disk model*, which is the unit disk equipped with the metric:
$$\frac{4(dx^2 + dy^2)}{(1 - (x^2 + y^2))^2} \ .$$
- In the *Poincaré half-plane model*, which is the half-plane $\{y > 0\}$ equipped with the metric $\frac{dx^2+dy^2}{y^2}$.

Isometries of the hyperbolic plane

The group of orientation-preserving isometries of $\mathbb{H}^2$ can be identified

- with $SO_0(2,1)$ (as visible in the quadric model),
- with $PSL(2,\mathbb{R})$, acting on the upper-half-space model by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} .z = \frac{az+b}{cz+d} \ .$$

Hyperbolic metrics and hyperbolic structures

Definition

A *hyperbolic metric* on S is a Riemannian metric of constant curvature -1.

Equivalently, a hyperbolic metric on S is a hyperbolic metric which is everywhere locally isometric to the hyperbolic plane. This leads to the definition of a hyperbolic structure: an atlas with charts in $\mathbb{H}^2$ and transition maps in $\mathrm{Isom}(\mathbb{H}^2)$.
The Riemann uniformization theorem provides an equivalence between conformal and hyperbolic structures.

Theorem (Poincaré-Riemann Uniformization)

Let S be a closed surface of genus at least 2, and let g be a Riemannian metric on S. There is a unique function $u : S \to \mathbb{R}$ such that $e^{2u}g$ is hyperbolic.

Back to Teichmüller space

Alternate definitions:

Definition

$\mathcal{T}_S$ is the quotient by $\mathrm{Diff}_0(S)$ of the space of conformal classes on S.

Definition

$\mathcal{T}_S$ is the quotient by $\mathrm{Diff}_0(S)$ of the space of hyperbolic metrics on S.

Key properties:

- $\mathcal{T}_S$ has finite dimension $6g - 6$,
- it is homeomorphic to a ball.

The Weil-Petersson metric

The most natural Riemannian metric on $\mathcal{T}_S$ – there are other natural distances.

Let $c \in \mathcal{T}_S$, with hyperbolic metric h, written in complex coordinates as

$$h = \rho|dz|^2 = \rho dz d\bar{z} \ .$$

Given $\dot{c} \in T_c\mathcal{T}_S$, it can be represented as a Beltrami differential $\mu = f\frac{d\bar{z}}{dz}$. There is a unique such representation by a *harmonic* Beltrami differential.

Definition

The WP scalar product is defined between two such tangent vectors:

$$\langle \dot{c}, \dot{c}' \rangle_{WP} = \int_S \mu \bar{\mu}' da_h \ ,$$

where μ and μ' are harmonic representatives.

Here da_h is the area form of the hyperbolic metric h.

The Fischer-Tromba metric, 1

Let now $h \in \mathcal{T}_S$ be a hyperbolic metric corresponding to a complex structure c, and let $\dot{h}$ be a tangent vector to $\mathcal{T}_S$ at h. This $\dot{h}$ only well-defined up to adding a trivial deformation, that is, a variation of the form

$$\dot{h}_v = \mathcal{L}_v h \ ,$$

where v is a vector field on S and $\mathcal{L}$ is the Lie derivative. However there is a unique way to "normalize" $\dot{h}$, by adding such a Lie derivative, in such a way that

$$\mathrm{div}(\dot{\mathrm{h}}) = 0 \ , \quad \mathrm{tr}_{\mathrm{h}}(\dot{\mathrm{h}}) = 0 \ .$$

The Fischer-Tromba metric, 2

Definition

The Fischer-Tromba metric is:

$$\langle \dot{c}, \dot{c}' \rangle_{FT} = \int_S \langle \dot{h}, \dot{h}' \rangle_h da_h \ ,$$

where

- $\dot{h}, \dot{h}'$ are supposed to be *normalized*,
- $\langle , \rangle_h$ denotes the extension to symmetric 2-tensors of the Riemannian metric h,
- da_h denotes the area form of h.

Lemma

The Weyl-Petersson and Fischer-Tromba metric are related by:

$$\langle \dot{c}, \dot{c}' \rangle_{WP} = \frac{1}{8} \langle \dot{c}, \dot{c}' \rangle_{FT} \ .$$

Key properties

- Kähler (Weil)

Recall that a Kähler manifold is a complex manifold M equiped with a symplectic structure ω such that $\omega(\cdot, i\cdot)$ is a Riemannian metric on M.

- negative sectional curvature (Tromba, Wolpert)
- non-complete (Wolpert) – metric completion is augmented Teichmüller space...
- ... quotient by mapping-class group is the Deligne-Mumford moduli space.
- geodesically convex (Wolpert)

The mapping-class group of S acts isometrically on $(\mathcal{T}_S, g_{WP})$ so the moduli space of S is equipped with the Weil-Petersson metric. The geometry of $(\mathcal{M}_S, g_{WP})$ is interesting.

The 3-dimensional hyperbolic space, 1

Definition

$\mathbb{H}^3$ is the unique complete, simply connected surface of constant curvature -1.

- As a quadric in the 4-dimensional Minkowski space $\mathbb{R}^{3,1}$, with the induced metric: $\{x \in \mathbb{R}^{3,1} \mid \langle x, x\rangle = -1 \ \wedge x_0 > 0\}$.
- In the *Poincaré ball model*, which is the unit disk equipped with the metric:
$$\frac{4(dx^2 + dy^2 + dz^2)}{(1 - (x^2 + y^2 + z^2))^2} .$$
- In the *Poincaré half-space model*, which is the half-space $\{z > 0\}$ equipped with the metric $\frac{dx^2+dy^2+dz^2}{z^2}$.

The 3-dimensional hyperbolic space, 2

Some properties:

- $\partial_\infty \mathbb{H}^3 = \mathbb{C}P^1$, with its standard conformal structure
- $\mathrm{Isom}_0(\mathbb{H}^3) = \mathrm{PSL}(2, \mathbb{C})$, acting on the upper half-space model
- $PSL(2, \mathbb{C})$ acts conformally on $\mathbb{C}P^1$...
- ... in fact by Möbius transformations.

Quasifuchsian hyperbolic manifolds

We now consider hyperbolic 3-manifolds – equipped with Riemannian metrics of constant curvature -1, and locally modelled on $\mathbb{H}^3$.

Definition

A hyperbolic manifold M is *quasi-Fuchsian* if it is homeomorphic to $S \times \mathbb{R}$, complete, and contains a non-empty compact geodesically convex subset.

Definition

$K \subset M$ is *geodesically convex* if any geodesic segment in M with endpoints in K is contained in K.

Extension to *convex co-compact* hyperbolic manifolds: complete, containing a non-empty compact geodesically convex subset.
The renormalized volume is a function defined on the space of those manifolds.

The conformal structure at infinity

Lemma

Let M be quasifuchsian. Then $M = \mathbb{H}^3/\rho(\pi_1(S))$, where $\rho(\pi_1(S))$ acts properly discontinuously on $\mathbb{H}^3 \cup (\partial_\infty \mathbb{H}^3 \setminus \Lambda_\rho)$.

Here Λ_ρ is the *limit set* of ρ: accumulation at infinity of any orbit of ρ.

Lemma

Λ_ρ is a quasi-circle*: the image of a circle under a quasi-conformal homeo of $\partial_\infty \mathbb{H}^3 = \mathbb{C}P^1$.*

Therefore $\partial_\infty \mathbb{H}^3 \setminus \Lambda_\rho = \Omega_+ \cup \Omega_-$.
Moreover, $\partial_\infty M = (\Omega_+/\rho(\pi_1 S)) \cup (\Omega_-/\rho(\pi_1 S))$.

Definition

The conformal structure at infinity $(c_-, c_+) \in \mathcal{T}_S \times \mathcal{T}_S$ on $\partial_\infty M$ are obtained by quotient.

The Bers Simultaneous Uniformization Theorem

Theorem (Bers)

The map $\mathcal{QF} \to \mathcal{T}_S \times \mathcal{T}_S$ sending a quasifuchsian hyperbolic metric to its conformal structures at infinity is a homeomorphism.

Ahlfors-Bers: extension to convex co-compact manifolds, with conformal structure on ∂M.
As a consequence, $V_R : \mathcal{QF} \to \mathbb{R}$ can be considered as a function $V_R : \mathcal{T}_S \times \mathcal{T}_S \to \mathbb{R}$.

Property 1 (Takhtajan-Teo 2003)

Let $c_- \in \mathcal{T}_S$. Then $V_R(c_-, \cdot) : \mathcal{T}_S \to \mathbb{R}$ is a Kähler potential for the g_{WP}.

That is, $\omega_{WP} = i\partial\bar{\partial} V_R(c_-, \cdot)$.

The volume of the convex core

Let M be quasifuchsian (resp. convex co-compact).

Lemma

The intersection of two closed non-empty geodesically convex subsets is non-empty and geodesically convex.

Definition

The convex core of M, $C(M)$, is its smallest non-empty closed geodesically convex subset.

Its volume is denoted by $V_C(M)$.
Through the Bers Theorem, $V_C : \mathcal{T}_S \times \mathcal{T}_S \to \mathbb{R}_{\geq 0}$.

V_C and the Weil-Petersson distance

Theorem (Brock 2003)

There exists $K > 1$ such that

$$\frac{1}{K} d_{WP}(c_-, c_+) - K \leq V_C(c_-, c_+) \leq K d_{WP}(c_-, c_+) + K \ .$$

No explicit control of K.

Consequences:

- on the geometry of hyperbolic mapping tori,
- on the large-scale WP geometry of moduli space,
- ...

V_C vs V_R

Property 2 (S.'03, Bridgeman-Canary '17)

Let M be quasifuchsian (resp. convex co-compact with incompressible boundary). There exists $C(\partial M)$ such that

$$V_C(M) - C \leq V_R(M) \leq V_C(M) + C \ .$$

Recently extended to all convex co-compact manifolds, possibly with compressible boundary (S.-Witten 2022).

Property 3 (S '13)

Let M be quasifuchsian, with S of genus g. Then

$$V_R(c_-, c_+) \leq 3\sqrt{\pi(g-1)}d_{WP}(c_-, c_+) + C(g) \ ,$$

So V_R is "coarsely equivalent" to V_C, but has additional properties (Kähler potential) and with better control (explicit upper bound).

The complex projective structure at infinity

The 4th property is a variational formula for V_R. Depends on the structure at infinity.

Definition

A *complex projective structure*, or $\mathbb{C}P^1$-structure, on a surface is an atlas with charts in $\mathbb{C}P^1$ and transition maps in $PSL(2,\mathbb{C})$.

If M is quasifuchsian (resp. convex co-compact) then each connected component of $\partial_\infty M$ is the quotient of an open subset of $\mathbb{C}P^1$ by a group. So equipped with a $\mathbb{C}P^1$-structure $\sigma \in \mathcal{CP}_{\partial M}$.

In the next slide we will "turn" σ into a holomorphic quadratic differential on $\partial_\infty M$.

The Schwarzian derivative

Definition

Let $\Omega \subset \mathbb{C}$ and $f : \Omega \to \mathbb{C}$ be holomorphic. It Schwarzian derivative is

$$\mathcal{S}(f) = \left(\left(\frac{f''}{f'} \right)' - \frac{1}{2} \left(\frac{f''}{f'} \right)^2 \right) dz^2 \ .$$

It is a *holomorphic quadratic differential* (HQD).

Two key properties:

1. $\mathcal{S}(f) = 0$ iff $f \in PSL(2, \mathbb{C})$.
2. $\mathcal{S}(f \circ g) = \mathcal{S}(g) + g^* \mathcal{S}(f)$.

Lemma

Let S be equipped with a CP^1-structure σ. Let $f : \tilde{S} \to \mathbb{D}$ be the Riemann uniformization map. Then $\mathcal{S}(f)$ can be defined in each chart, and it does not depend on the chart.

"$\mathcal{S}(f) = \sigma_F - \sigma$".

The Schwarzian at infinity

Definition

$\mathcal{CP}_S$ is the space of complex projective structures on S, up to isotopy.

Corollary

$\mathcal{CP}_S$ is topologically a ball of dim $12g-12$. It is an affine bundle over $\mathcal{T}_S$, with fiber HQD_c over c. Canonical identification with $T^*\mathcal{T}_S$.

Definition

The *Schwarzian derivative at infinity* q of M is defined, for each boundary component $\partial_i M$ of M, as $q = -\mathcal{S}(f)$, where $f : \widetilde{\partial_i M} \to \mathbb{D}$ is the uniformization map.

The variational formula V_R

Recall that HQD are naturally identified elements of the (complexified) $T^*\mathcal{T}_S$ through their duality with Beltrami differentials: if $\mu = f\frac{d\bar{z}}{dz}$ and $q = g(z)dz^2$ then

$$\langle \mu, q\rangle = \int_S \mu q = \int_S fgdzd\bar{z} \in \mathbb{C} ~.$$

Property 4

Let $c \in \mathcal{T}_{\partial M}$ and let $\dot{c} \in T_c\mathcal{T}_{\partial M}$.

$$(d_c V_R)(\dot{c}) = \mathrm{Re}(\langle \mathrm{q}, \dot{\mathrm{c}}\rangle) ~.$$