Research Statement

My main research interest is in geometric topology and specifically, the study of 3-dimensional manifolds and the interplay between their topology and the geometric structures that they admit.

In three dimensions Thurston's Geometrization [Thu82], whose last case was resolved by Perelman in 2003 [Per03a, Per03b, Per03c], states that every compact 3-manifold can be cut into finitely many pieces, each of which admits, in its interior, one of eight homogeneous geometric structures. Of these eight geometries, seven are supported on 3-manifolds that admit a complete topological classification. The last type of geometry, hyperbolic geometry, is supported on manifolds that, to date, do not have a complete topological classification.

The Geometrization Theorem also prescribes conditions for a compact 3-manifold M so that its interior $\mathrm{int}(M)$ admits a complete hyperbolic structure, a Riemannian metric with constant sectional curvature -1. The topological restrictions amount to being irreducible (2-spheres bound 3-balls), atoroidal (π_1-injective tori are homotopic into ∂M) and with infinite fundamental group. The Tameness Theorem [Ago04, CG06] states that if M is a hyperbolic 3-manifold with finitely generated fundamental group, then M is the interior of a compact 3-manifold $\overline{M}$.

By combining the Geometrization and the Tameness Theorem one obtains that if M is hyperbolic with finitely generated fundamental group, then M is the interior of a compact, irreducible and atoroidal 3-manifold $\overline{M}$ with infinite fundamental group. Therefore, we have a topological *characterisation* of finite-type hyperbolic 3-manifolds. Moreover, these conditions are not too restrictive, and in some sense, they are *generic* in the set of compact 3-manifolds.

The papers I wrote can be subdivided into two main areas. I have been extending topological characterisations of hyperbolic 3-manifolds when the ambient manifold is not compact. By the previous characterisation we only need to consider infinite type 3-manifolds, that is, manifolds that have infinitely generated fundamental group. My main result in this area is a complete topological characterisation of hyperbolic manifolds in a large class of infinite-type 3-manifolds, see Theorem 5.

The second area falls into what is referred to as 'effective geometrization', whose aim is to understand the hyperbolic structure guaranteed by Geometrization. Specifically, I have been studying the geometry of certain knot complements in Seifert-fibered spaces by understanding their volume functions. For large classes of these knots, with my collaborators we gave a hyperbolization result, the first quasi-isometric bounds and general asymptotics for their volume, see Theorem 11, 12, and 13.

The second project led me to study knots $\widehat{\gamma}$ in the unit tangent bundle $PT(S)$, for S a hyperbolic surface, that correspond to periodic orbits of the geodesic flow. Properties of the geodesic flow, such as ergodicity, translate to the fact that for 'generic' geodesics γ_L of length L the corresponding volume complement $PT(S) \setminus \widehat{\gamma}_L$ blows-up as L goes to infinity. To further explore this relationship, I am interested in learning more about the dynamical properties of the geodesic flow and how they relate to the distribution of its closed orbits. Another notion of genericity is achieved by fixing a generating set $\mathcal{S}$ of $\pi_1(S)$ and interpreting a closed geodesic as a 'random' word in $\pi_1(S)$. One might be able to use the theory of random walks to show that the volume grows 'almost surely' linearly with respect to word-length, see Question 5 in Section 2.2.

The rest of the Research Statement is organised as follows. In Section 1, I will describe my work to date. In Section 2, I will describe some new directions of research.

1. **Past work**

The first sub-section will focus on my geometrization results for infinite-type 3-manifolds and some other results regarding these 3-manifolds. In the second sub-section, I will describe the quasi-isometric results on volume bounds for certain knots in Seifert-fibered spaces.

1.1. Part I: Geometric Structures on infinite-type 3-manifolds

Except for specific examples, not much is known, to date, on hyperbolic 3-manifolds with non-finitely generated fundamental group. One of the first such examples appears in [BO88] where the authors build infinite-type hyperbolic 3-manifolds to show that hyperbolic 3-manifolds have no uniform lower bound on injectivity radius. In [Thu98a] Thurston builds infinite-type hyperbolic 3-manifolds as geometric limits of quasi-Fuchsian manifolds. In [BMNS16], the authors build hyperbolic 3-manifolds by glueing together, via sufficiently complicated maps, collections of hyperbolic 3-manifolds coming from a finite list. The glueing patterns that are allowed can contain infinitely many manifolds yielding hyperbolic 3-manifolds with non-finitely generated fundamental groups. Finally, in [SS13] the authors build the examples as glueing of *acylindrical* hyperbolisable 3-manifolds with restrictions on the topology of their boundary components. Most of my work has been into extending these type of arguments to 3-manifolds admitting cylinders.

In general, given a 3-manifold M, there are immediate obstructions to the existence of a complete hyperbolic metric in its interior. Indeed, a hyperbolisable 3-manifold M is homeomorphic to $\mathbb{H}^3/\Gamma$, with $\Gamma \leqslant PSL_2(\mathbb{C})$ a torsion-free Kleinian group, where Γ is isomorphic to $\pi_1(M)$. Since Γ is a discrete group, no $\gamma \in \Gamma$ has infinitely many roots[1], see [Fri11]. Covers of hyperbolisable manifolds are hyperbolisable as well, and we say that a manifold M is *locally hyperbolic* if every cover $N \twoheadrightarrow M$ with $\pi_1(N)$ finitely generated is hyperbolisable.

In [Cre18] we answer, in the negative, the following question of Agol [DHM06, Mar07]:

Question (Agol)**.** If a 3-manifold M has no divisible subgroups in $\pi_1(M)$ and is locally hyperbolic is M hyperbolic?

Theorem 1 (Cremaschi, [Cre18])**.** There exists a locally hyperbolic 3-manifold M_∞ with no infinitely divisible elements in its fundamental group that does not admit any complete hyperbolic metric.

The manifold M_∞, see Figure 1, has the following topological properties:

(i) it admits an exhaustion by nested compact hyperbolisable 3-manifolds $\{M_n\}_{n\in\mathbb{N}}$;
(ii) for all $n \in \mathbb{N}$, the sub-manifold M_n has incompressible boundary in M_∞ so that $\pi_1(M_n)$ injects into $\pi_1(M_{n+1})$;
(iii) each component S of ∂M_n has uniformly bounded genus.

The class $\mathcal{M}^B$ of 3-manifolds satisfying (i)-(iii) contains both our example M_∞ and the hyperbolisable complement of a Cantor set in $\mathbb{S}^3$ constructed in [SS13].

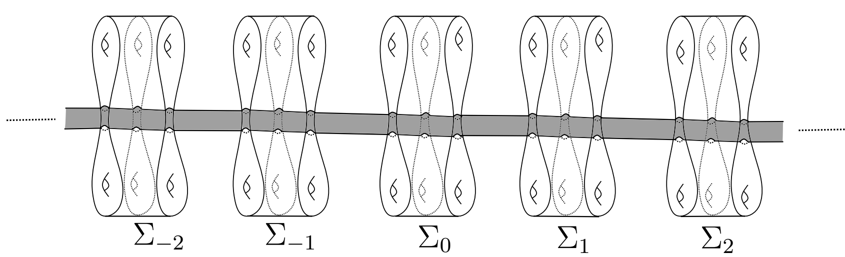

FIGURE 1. The manifold M_∞, where the thickened components around the surfaces Σ_i's are joined by thickened annuli. The bi-infinite annulus A is shaded.

The obstruction to the hyperbolicity of M_∞ is the properly embedded bi-infinite cylinder $A \hookrightarrow M_\infty$. Note that, both ends of A *witness* an 'infinite amount of topology' in M_∞. This

[1] A root of γ is $a \in \pi_1(M)$ such that $\gamma = a^n$ for some $n \in \mathbb{N}$.

phenomenon is realised by the collection of embedded surfaces $\{\Sigma_i\}_{i\in\mathbb{Z}}$ exiting both ends of the cylinder, see Figure 1. The collection of surfaces $\{\Sigma_i\}_{i\in\mathbb{Z}}$ is properly embedded in M_∞, each Σ_i is incompressible, and $A\cap\Sigma_i$, $A\cap\Sigma_j$ induce the only loops in Σ_i, Σ_j respectively that are homotopic in M_∞. One can show, see [Cre18], that if M_∞ was hyperbolic, then the soul γ of the cylinder A must be represented by a parabolic element. Using this observation and the theory of simplicial hyperbolic surfaces, developed by Thurston, Bonahon and Canary [Thu78, Can96, Bon86], this forces the surfaces $\{\Sigma_i\}_{i\in\mathbb{N}}$ to have homotopic loops $\alpha_i \neq \gamma$ in M_∞ giving a contradiction with the topology of M_∞. The fact that the Σ_i's have bounded topological type is needed to obtain area bounds for some geometric representatives of the Σ_i's.

The above example shows that a locally hyperbolic 3-manifold M without infinitely divisible elements in its fundamental group does not necessarily admit a complete hyperbolic metric. However, the example constructed in [Cre18] is homotopy equivalent to a hyperbolic 3-manifold. Therefore, the obstruction given by the configuration in Figure 1 is not a homotopy invariant. However, a slight modification of [Cre18] gives an example, see [Cre20], that is not homotopy equivalent to any hyperbolic 3-manifold.

Theorem 2 (Cremaschi, [Cre20])**.** There are locally hyperbolic 3-manifolds without infinitely divisible elements in π_1 that are not homotopy equivalent to any hyperbolisable 3-manifold.

Building on [Cre18, Cre20] the main Theorem of my doctoral Thesis, see [Cre19], is a complete topological characterisation, in the spirit of Thurston's hyperbolisation, of hyperbolisable manifolds in $\mathcal{M}^B$. The only obstruction is given, as in M_∞, by having a properly embedded bi-infinite annulus A such that each end of A witnesses an infinite amount of topology.

To carefully study these annuli, I introduce, for manifolds M in $\mathcal{M}^B$, a notion of manifold *bordification* which we will denote by $\overline{M}$. The bordification $\overline{M}$ is constructed as follows: given $M\in\mathcal{M}^B$ we take a *maximal* collection of pairwise disjoint properly embedded products $F\times[0,\infty)\hookrightarrow M$, for F a π_1-injective surface, and then, for each such product we add to M a copy of $\text{int}(F)\times\{\infty\}$. The result is the *canonical maximal* bordified manifold $(\overline{M},\partial\overline{M})$ of $M\in\mathcal{M}^B$. The manifold $\overline{M}$ is a 3-manifold with boundary whose interior is homeomorphic to M. In general, each component of $\partial\overline{M}$ is a *non-compact* finite-type punctured surface. The bordification $\overline{M}$ only depends on the topology of M, and any two maximal bordifications for M are homeomorphic. Using the maximal bordification, I extend two remarkable results in the theory of 3-manifold topology, [Joh79, Jac80, Wal68], to the infinite-type setting:

Theorem 3 (Cremaschi, [Cre19])**.** Given $M\in\mathcal{M}^B$ and its maximal bordification $\overline{M}$ then, there exists a unique, up to proper isotopy, characteristic sub-manifold $(N,R)\hookrightarrow(\overline{M},\partial\overline{M})$. Moreover, each component of N admits a manifold compactification to either a compact I-bundle or a solid torus.

and

Theorem 4 (Cremaschi, [Cre19])**.** Let $M,N\in\mathcal{M}^B$ and $\varphi: M\to N$ be a homotopy equivalence. Then, if the maximal bordification $\overline{M}$ is acylindrical the map φ is homotopic to a homeomorphism.

A *doubly peripheral cylinder* in $(\overline{M},\partial\overline{M})$ is an essential cylinder $(C,\partial C)\subseteq(\overline{M},\partial\overline{M})$ whose boundaries are both peripheral in $\partial\overline{M}$, see Figure 2. Using the techniques of [Cre20], one can show that if $M\in\mathcal{M}^B$ has a doubly peripheral cylinder, then M is not hyperbolic.

We say that a manifold $(M,\mathcal{P})$ is *pared* if $\mathcal{P}\subseteq\partial M$ contains all tori components of ∂M, the other components of $\mathcal{P}$ are essential π_1-injective annuli in ∂M and M is acylindrical relative to $\mathcal{P}$. Then, we let $AH(M,\mathcal{P})$ denote the set of uniformisations $\Gamma\subseteq\text{PSL}_2(\mathbb{C})$ of $M=\mathbb{H}^3/\Gamma$ in which parabolics represent the components of $\mathcal{P}$, $\mathcal{P}$ is called the *parabolic locus*. By Theorem 3 it is not hard to see how if M has no doubly peripheral annuli, one can pick a pairing structure $\mathcal{P}$ for $\overline{M}$. Then, given $M=\{M_i\}_{i\in\mathbb{N}}\in\mathcal{M}^B$ with a pairing structure $\mathcal{P}$ on the bordification $\overline{M}$

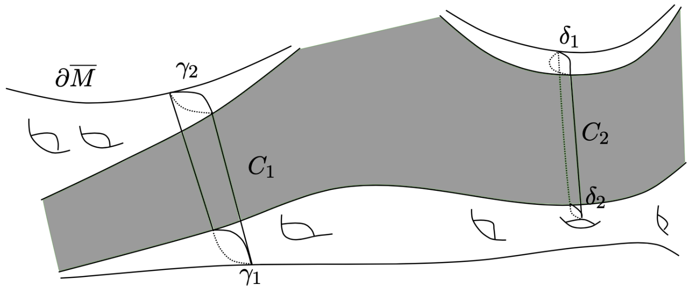

FIGURE 2. The loops γ_1, γ_2 and δ_1 are peripheral in $\partial\overline{M}$ hence, C_1 is a doubly peripheral annulus while C_2 is not, M is shaded.

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one can induce well-defined pairing structures $\mathcal{P}_i$ on each M_i. Hence, by picking a collection of hyperbolic structures $\Gamma_i \in AH(M_i, \mathcal{P}_i)$ and a compactness argument of Thurston, see [Thu98b], one can build a hyperbolic 3-manifold N that is homotopy equivalent to $(\overline{M}, \mathcal{P})$ and such that the image, under the homotopy equivalence, of $\mathcal{P}$ are parabolic elements in N. Then, by using Theorem 4 and building on the techniques of [Cre20] one obtains:

Theorem 5 (Cremaschi, [Cre19]). *Let $M \in \mathcal{M}^B$, then M is homeomorphic to a complete hyperbolic 3-manifold if and only if the maximal bordified manifold $\overline{M}$ does not admit any doubly peripheral cylinder.*

On a separate note, in collaboration with Souto, see [CS18], we constructed two remarkable examples of discrete, non finitely generated subgroups of $\mathrm{PSL}_2(\mathbb{C})$ that show how algebraic properties that hold for discrete finitely generated subgroups fail for non-finitely generated groups. In particular, we show that there exists a non-torsion-free Kleinian group Γ such that Γ has no non-trivial finite quotients and a torsion-free Kleinian group Λ that is not residually finite.

Lastly, a Theorem of Souto–Purcell [PS10] states that any hyperbolisable one-ended 3-manifold that embeds in the 3-sphere is the geometric limit of knot complements. With Franco Vargas-Pallete, we showed the following similar result:

Theorem 6 (Cremaschi–Vargas Pallete, [CVP20]). *Let $M = \mathbb{H}^3/\Gamma$ be a hyperbolic 3-manifold, not necessarily of finite type, without rank two cusps, $M = \cup_{i\in\mathbb{N}} M_i$ and M_i is π_1-injective in M. If M admits an embedding $\iota : M \hookrightarrow \mathbb{S}^3$ then there exists a sequence of Cantor sets $\mathcal{C}_i \subseteq \mathbb{S}^3$, $i \in \mathbb{N}$, such that:*

(i) *$N_i \doteq \mathbb{S}^3 \setminus \mathcal{C}_i$ is hyperbolic;*
(ii) *the N_i's converge geometrically to M.*

Remark 7. The above Cantor set complements do not lie in $\mathcal{M}^B$.

1.2. Hyperbolic Knot complements in Seifert-Fibered Spaces

Given a surface S with $\chi(S) < 0$ let $M \doteq PT(S)$ be the projective unit tangent bundle of S and consider knots K that arise as lifts $\overline{\gamma}$ of loops $\gamma \subseteq S$, and we will refer to these as *topological lifts*. A special class of such lifts corresponds to closed orbits of the geodesic flow on $PT(S)$, we will also refer to these as *canonical lifts* and denote them by $\widehat{\gamma}$. For loops intersecting minimally (i.e. in minimal position), or geodesics, there is a simple criterion to decide whether their canonical lift has hyperbolic complement:

Theorem 8 (Foulon–Hasselblatt, [FH13]). *Let γ be a primitive closed geodesic on a hyperbolic surface Σ. Then, $M_{\widehat{\gamma}}$ admits a finite volume complete hyperbolic metric if and only if γ is filling.*

The curve γ being filling is an obvious necessary condition since if not we can find a simple closed curve $\eta \hookrightarrow S$ such that $\iota(\gamma, \eta) = 0$. Then, the full pre-image $p^{-1}(\eta)$ of η under the bundle map $p : PT(S) \to S$ gives an essential torus T in $M_{\widehat{\gamma}}$ that is not homotopic into the boundary of a regular neighbourhood $N_\varepsilon(\widehat{\gamma})$ of $\widehat{\gamma}$. Once the criterion of hyperbolicity is settled, one can look for bounds of the volume related to properties of the geodesic. In this direction, the best-known bounds are given by the following results:

Theorem 9 (Bergeron–Pinsky–Silberman, [BPS19])**.** Given a hyperbolic metric X on a surface S, there is a constant $C_X > 0$ such that for any finite set γ of primitive periodic geodesics filling X we have:

$$\mathrm{Vol}(M_{\widehat{\gamma}}) \leq C_X \ell_X(\gamma)$$

Thus, the upper bound depends on geometric quantities associated to γ. On the other hand, the lower bound is purely topological:

Theorem 10 (Rodríguez-Migueles, [RM20])**.** Given a pants decomposition $P = \{P_i\}_{i=1}^{-\chi(S)}$ on a surface S, with $\chi(S) < 0$, and a filling geodesic $\gamma \subseteq S$, then:

$$\frac{v_3}{2} \sum_{i=1}^{-\chi(S)} |\{\text{homotopy classes of the arcs } \gamma \cap P_i\}| \leq \mathrm{Vol}(M_{\widehat{\gamma}})$$

for v_3 the volume of the regular ideal tetrahedron.

However, neither one of these bound is *tight*. For example, if $\gamma = \varphi(\eta)$ for $\varphi \in Mod(S)$ and η a filling geodesic, the canonical lift complements $M_{\widehat{\gamma}}$ and $M_{\widehat{\eta}}$ are homeomorphic and, by Mostow rigidity, they have the same volume. If one takes a Dehn Twist φ along some curve α then, for a given hyperbolic metric X on S the family of filling geodesics $\gamma_n \doteq \varphi^n(\eta)$ has the property that:

$$\lim_{n\to\infty} \ell_X(\gamma_n) = \infty.$$

Nevertheless, the volumes of the $M_{\widehat{\gamma}_n}$ stay constant. Similarly, one can build examples in which $\iota(\gamma_n, \gamma_n)$ goes to infinity but $\mathrm{Vol}(M_{\widehat{\gamma}_n}) < V$. Moreover, one can also build more interesting examples in which the manifolds $M_{\widehat{\gamma}}$ are not homeomorphic, see [RM20].

The projective tangent bundle $PT(S)$ is a circle bundle on S, and so it is fibered by circles. In general, a compact irreducible 3-manifold M is said to be a *Seifert-fibered space* if it can be written as a disjoint union of circles satisfying specific local properties. Given any Seifert-fibered space M by collapsing the circles one obtains a natural projection map $p : M \to \mathcal{O}$ where $\mathcal{O}$ is, in general, a 2-orbifold. Going forward, $\mathcal{O}$ will always denote a *good orbifold*, i.e. an orbifold that admits a finite index orbifold cover that is a surface. We also define a geodesic γ in $\mathcal{O}$ to be a *filling geodesic* if it is filling on the surface S defined to be the orbifold $\mathcal{O}$ minus its singular cone points. Working with Rodríguez-Migueles, I extended Theorem 8 and 10 to this class of 3-manifolds:

Theorem 11 (Cremaschi–Rodríguez-Migueles, [CRM20])**.** Suppose $\mathcal{O}$ is a hyperbolic 2-orbifold and γ a filling multi-curve of primitive closed geodesics in $\mathcal{O}$. Then, for any topological lift $\overline{\gamma}$ in a Seifert-fibered manifold M over $\mathcal{O}$ the complement $M_{\overline{\gamma}} \doteq M \setminus \overline{\gamma}$ is a hyperbolic manifold of finite volume. Moreover, for any pants decomposition Π of $\mathcal{O} \setminus \mathrm{cone}(\mathcal{O})$ we have:

$$\mathrm{Vol}(M_{\overline{\gamma}}) \geq \frac{v_3}{2} \sum_{P\in\Pi} (\sharp\{\text{ isotopy classes of } \overline{\gamma}\text{-arcs in } p^{-1}(P)\} - 3).$$

In the case of a link $\widehat{\Gamma} \subseteq PT(S)$ such that the projection $\Gamma \subseteq S$ is a collection of simple closed curves in minimal position Theorem 9 still holds, however, Theorem 10 gives no information since it is uniformly bounded by a constant depending on $\chi(S)$. To address this issue, with Yarmola and Rodríguez-Migueles, we gave the first quasi-isometric bounds in the particular case where Γ is a filling pair of simple closed curves:

Theorem 12 (Cremaschi–Rodríguez-Migueles–Yarmola, [CRMY19]). Let S be a hyperbolic surface, $N = PT(S)$, and let (α, β) be a filling pair of essential simple closed curves on S in minimal position. Then:

$$\mathrm{Vol}(N_{(\widehat{\alpha},\widehat{\beta})}) \asymp_S \inf_{P_\alpha, P_\beta} d_{\mathcal{P}(S)}(P_\alpha, P_\beta) \asymp_S d_{WP}(\mathcal{T}_\alpha, \mathcal{T}_\beta)$$

where P_α, P_β are any pants decompositions of S with $\alpha \subseteq P_\alpha, \beta \subseteq P_\beta$ and $\asymp_S$ denotes a quasi-isometry with constant depending only on S. Lastly, $d_{WP}(\mathcal{T}_\alpha, \mathcal{T}_\beta)$ is the Weil-Petersson distance between the strata corresponding to α and β.

More generally, Theorem 12 works for hyperbolic links $\overline{\Gamma}$ in a Seifert-fibered space N over S when $\overline{\Gamma}$ satisfies specific properties. If N admits a fibration by surfaces carrying the components of $\overline{\Gamma}$ then, $\overline{\Gamma}$ is defined as a *stratified link*. The collection $\left\{(\overline{\Gamma}_i, S_i)\right\}$ of fibres S_i and components $\overline{\Gamma}_i$ of $\overline{\Gamma}$ is called a *stratification* of $\overline{\Gamma}$. In this setting, we obtain the following bounds:

Theorem 13 (Cremaschi–Rodríguez-Migueles–Yarmola, [CRMY19]). Let N be a Seifert-fibered space over a hyperbolic surface S and $\overline{\Gamma} \subseteq N$ be a hyperbolic link. If $\mathcal{H} = \{(\overline{\Gamma}_i, S_i)\}_{i=1}^n$ is a stratification of $\overline{\Gamma}$, then there exist constants $K_1 > 1$ and $K_0 > 0$, depending only on $\kappa(\mathcal{H})$, such that:

$$\frac{1}{nK_1} \inf_{\mathfrak{P}} \left(\sum_{i=1}^n d_{\mathcal{P}(S_{\mathcal{H}})}(P_i, P_{i+1}) \right) \leq \mathrm{Vol}(N_{\overline{\Gamma}}) \leq K_1 \inf_{\mathfrak{P}} \left(\sum_{i=1}^n d_{\mathcal{P}(S_{\mathcal{H}})}(P_i, P_{i+1}) \right) + nK_0$$

where $\mathfrak{P} = \{P_i\}_{i=1}^n$ is any collection of pants decompositions with $\Gamma_i^{\mathcal{H}} \subseteq P_i$ and we let $P_{n+1} = \psi_{\mathcal{H}}(P_1)$.

2. Future Projects

2.1. Space of hyperbolic structures on infinite-type hyperbolic 3-manifolds

Having answered the question of hyperbolicity for a large class of 3-manifolds, I want to study the space of hyperbolic structures and see which results extend from the rich deformation theory of finite-type hyperbolic manifolds. In the proof of Theorem 5, the hyperbolic structure on $(M, \mathcal{P})$ came by randomly choosing hyperbolic structures $\Gamma_i \in AH(M_i, \mathcal{P}_i)$ and using compactness arguments. By carefully picking the Γ_i's, one can show:

Theorem 14. Let $M \in \mathcal{M}^B$ be hyperbolisable, $\overline{M}$ be the maximal bordification with pairing structure $\mathcal{P}$, and let $\coprod_{i\in\mathbb{N}} S_i \doteq \partial\overline{M} \setminus \mathcal{P}$. Then, for any collection of end invariants $\{\lambda_i\}_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} E(S_i)$ there is $\Gamma \in AH(M, \mathcal{P})$ such that the end facing S_i has the structure given by λ_i.

By Theorem 14 for any sequence $([X_i])_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} \mathcal{T}(S_i)$ there exists a hyperbolic manifold $N = \mathbb{H}^3/\Gamma$ homeomorphic to $M = \cup_{i\in\mathbb{N}} M_i \in \mathcal{M}^B$ whose conformal boundary is $\partial_c\Gamma = \coprod_{i=1}^\infty X_i$. Because each tame end of N is geometrically finite, [Can96], we say that such a manifold has *geometrically finite product structures* and we call the space of such manifold $GFP(M, \mathcal{P})$. Since, for $M = \cup_{i\in\mathbb{N}} M_i \in GFP(M, \mathcal{P})$ the covers corresponding to $\pi_1(M_i)$ are geometrically finite one should think of $GFP(M, \mathcal{P})$ as the analogue to the space of geometrically finite structures on a finite-type hyperbolic manifold.

If $\partial\overline{M} \setminus \mathcal{P}$ has infinitely many components $\{S_i\}_{i\in\mathbb{N}}$ that are not triply punctured spheres, and Γ, Γ' induce $([X_i])_{i\in\mathbb{N}}$ and $([Y_i])_{i\in\mathbb{N}}$ on $\{S_i\}_{i\in\mathbb{N}}$ respectively with $d_T([X_i], [Y_i]) = i$, for d_T the Teichmüller metric then, they cannot be quasi-conformally conjugate. Since any quasi-conformal map has bounded dilatation, the conformal structures corresponding to a quasi-conformal conjugation of Γ are a bounded distance away; thus Γ' cannot be quasi-conformally conjugate to Γ. This, non-surprisingly, is the first significant deviation from the finite-type setting in which $GF(M, \mathcal{P})$ was connected. Then, given Γ with conformal structures $([X_i])_{i\in\mathbb{N}}$ by a quasi-conformal deformation of Γ one can only get sequences $([Y_i])_{i\in\mathbb{N}}$ such that $\sup_{i\in\mathbb{N}} d_T([X_i], [Y_i]) < K$ for some uniform K. Thus, we define:

Definition 15. Given $M \in \mathcal{M}^B$ and $\Gamma \in AH(M,\mathcal{P})$ with $M = \mathbb{H}^3/\Gamma$. Let $([X_i])_{i\in I}$ be the conformal structures corresponding to $\{S_i\}_{i\in I} \subseteq \pi_0(\partial\overline{M} \setminus \mathcal{P})$ and define $\mathcal{T}_{\mathrm{b}}(\partial_c\Gamma)$ to be the subset of $\prod_{i\in I} \mathcal{T}(S_i)$ given by points $([Y_i])_{i\in I}$ such that $\sup_{i\in I} d_T([X_i],[Y_i]) < \infty$ and topologise $\mathcal{T}_{\mathrm{b}}(\partial_c\Gamma)$ by the sup-metric.

Note that, the set I is possibly infinite and we are allowing $M = \mathbb{H}^3/\Gamma$ to have simply degenerate ends in $(\partial\overline{M} \setminus \mathcal{P}) \setminus (\coprod_{i\in I} S_i)$. Then, the best result that one can hope to show is:

Question 1. Given $(M,\mathcal{P}) \in \mathcal{M}^B$ and a uniformisation $\Gamma \in AH(M,\mathcal{P})$ do we have:

$$QC(\Gamma) = \mathcal{T}_{\mathrm{b}}(\partial_c\Gamma)$$

By mimicking the finite-type proof strategy, one of the critical ingredients of the proof is to show that given $\Gamma \in AH(M,\mathcal{P})$ there is no quasi-conformal deformation of Γ whose support is contained in the limit set Λ_Γ. Any such group is called *quasi-conformally rigid* on its limit set.

Question 2. Let $\Gamma = \cup_{i\in\mathbb{N}}\Gamma_i$ be a Kleinian group with $\Gamma_i \subsetneq \Gamma_{i+1}$. If $M \doteq \mathbb{H}^3/\Gamma$ and $M \in \mathcal{M}^B$ with exhaustion given by $M = \cup_{i\in\mathbb{N}} M_i$ and $M_i = \mathbb{H}^3/\Gamma_i$ then is Γ quasi-conformally rigid on its limit set Λ_Γ?

The idea to prove Question 2 is to modify an argument due to McMullen [McM96]. In [McM96] the author shows that a hyperbolic 3-manifold M with bounded injectivity radius in the convex core is quasi-conformally rigid. The manifolds in Question 2 do not satisfy the injectivity radius conditions, which we can think of as a condition locally bounding the geometry. However, they have infinitely many surfaces, going out every end, with uniformly bounded genus which can be used to achieve local geometric bounds. I want to use the local bounds provided by these surfaces to recover McMullen's result. To complete Question 1, one needs to check that the natural maps coming from the Theory of Beltrami-differential, see [Hub06], extends. However, this should not represent an issue since the analytic theory does not care about the groups being non-finitely generated.

Question 3. If $M \in \mathcal{M}^B$ admits a hyperbolic structure and $\partial\overline{M} \setminus \mathcal{P} = \varnothing$, is the hyperbolic structure unique?

To avoid talking about relatively acylindrical manifolds, we will assume that the gaps $U_i = \overline{M_i \setminus M_{i-1}}$ are acylindrical, which implies $\partial\overline{M} = \varnothing$. Then, if we have a hyperbolic structure Γ on M it must have full limit set: $\Lambda_\Gamma = \mathbb{S}^2$ and by Question 2 we know that there are no quasi-conformal deformations of Γ. However, there could still be other structures Γ' that are not quasi-conformally equivalent to Γ. By using the structures given by the covering of $M = \mathbb{H}^3/\Gamma$ corresponding to the fundamental groups of M_i or $M_j \setminus M_i$ we obtain natural maps:

$$\varphi_i : AH(M) \to AH(M_i) \qquad \varphi_{j,i} : AH(M) \to AH(\overline{M_j \setminus M_i}), \quad j > i$$

such that for $j > i$: $\varphi_i = \psi_{j,i} \circ \varphi_j$ for $\psi_{j,i}$ the natural restriction map induced by: $M_i \hookrightarrow M_j$. Namely, one can think of an element of $AH(M)$ as a sequence of compatible elements of $AH(M_i)$. Moreover, in the case that the M_i's are acylindrical, we have that the image of φ_i is, by the Covering Theorem [Thu78, Can96], in $GF(M_i) \subseteq AH(M_i)$. If for all i the image of φ_i were to consist of a point we would prove Question 3. Studying these maps and relating them to the skinning map[2] might provide useful insights to prove Question 3. An important open conjecture related to these topics is:

Question (Minsky)**.** Given an acylindrical hyperbolic finite type 3-manifold is the diameter of the skinning map bounded by a constant C only depending on the maximal genus of the components of ∂M.

[2]Important work on the skinning map has been done by Kent, Dumas, Minsky, Bromberg and others [DK09, Ken10, KM14, Dum15, BMK18].

If Minsky's question 2.1, were to be proven then, an argument involving the skinning map would automatically imply Question 3.

2.2. **Finite volume**

In what follows we will use $\overline{\gamma}$ for a general topological lift and $\widehat{\gamma}$ for the canonical lift. I propose to study the geometry of these knots complements in projective tangent bundles further.

Question 4. Given a collection of filling geodesics $\{\gamma_n\}_{n\in\mathbb{N}}$ on S is there a criterion depending only on the $\{\gamma_n\}_{n\in\mathbb{N}}$ such that $\limsup \mathrm{Vol}(M_{\widehat{\gamma}_n}) < \infty$?

By the examples constructed in [RM20], one can construct sequences $\{\Gamma_n\}_{n\in\mathbb{N}} \subseteq S$ with the Γ_n collections of closed loops in minimal position such that $\limsup \mathrm{Vol}(M_{\overline{\Gamma}_n}) < \infty$ for $\overline{\Gamma}_n$ any topological lift of Γ_n. These examples essentially come by doing annular Dehn twist, on say $\overline{\Gamma}_1$, in M. I suspect that this is 'essentially' the unique construction producing bounded volume sequences. We define an *immersed Dehn twist* of γ along a minimal position, not necessarily prime, loop $\alpha \subseteq S$ in the following way. Assume that $\alpha\cap\gamma \neq \varnothing$ and pick $x_1,\dots,x_k \in \alpha\cap\gamma$ then we define the *immersed Dehn twist* $D^n_{\alpha,\vec{x}}(\gamma)$ by cutting γ at $\vec{x} = \{x_1,\dots,x_k\}$ and following α n-times before closing up. If $\vec{\alpha} = \{\alpha_1,\dots,\alpha_m\}$ has m components this can be generalised by choosing how many times $\vec{n} = \{n_1,\dots,n_k\}$ to push along each component, in which case we write $D^{\vec{n}}_{\vec{\alpha},\vec{x}}(\gamma)$.

For two lifts $\overline{\Gamma}_1, \overline{\Gamma}_2 \subseteq PT(S)$ of $\Gamma_1, \Gamma_2 \subseteq S$ we say that they are immersed Dehn twist of each other if $\Gamma_2 = D^{\vec{n}}_{\alpha,\vec{x}}(\Gamma_1)$ for some $\vec{n}, \vec{x}$ and $\vec{\alpha}$. Note that any two links are immersed Dehn twist of each other, however, the loops $\vec{\alpha}$ usually depend on both Γ_1 and Γ_2. Moreover, if $\overline{\Gamma}_1, \overline{\Gamma}_2 \subseteq PT(S)$ are immersed Dehn twist of each other, then $\overline{\Gamma}_2$ can be obtained by $\overline{\Gamma}_1$ by doing some annular Dehn twist surgery. Therefore, all links obtained from Γ_1 via immersed Dehn twist along a <u>fixed</u> collection α have uniformly bounded volume, the statement I believe to hold is:

Theorem 16. Let $\{\overline{\Gamma}_n\}_{n\in\mathbb{N}} \subseteq M = PT(S)$ be topological lifts of collections of minimal positions loops $\{\Gamma_n\}_{n\in\mathbb{N}} \subseteq S$. If $\limsup_n \mathrm{Vol}(M_{\overline{\Gamma}_n}) < V$ then, up to sub-sequence which we still denote by $\{\Gamma_n\}_{n\in\mathbb{N}}$, there exists a collection of loops $\Delta \subseteq S$ and lifts $\overline{\Delta}$ such that:

- the $\{\Gamma_n\}_{n\in\mathbb{N}}$ are obtained by doing immersed Dehn twist along Δ of Γ_1;
- the lifts $\{\overline{\Gamma}_n\}_{n\in\mathbb{N}}$ are obtained by doing annular Dehn twist of $\overline{\Gamma}_1$ along parallel copies of $\overline{\Delta}$.

which gives the following corollary:

Corollary 17. Let $\{\overline{\Gamma}_n\}_{n\in\mathbb{N}} \subseteq PT(S)$ be lifts of $\Gamma_n \subseteq S$, for $n\in\mathbb{N}$. If for large enough n the Γ_n's are not immersed Dehn twist of Γ_k, for some $k\in\mathbb{N}$, then $\limsup \mathrm{Vol}(M_{\Gamma_n}) = \infty$.

Question 5. Given a *generic filling geodesic* γ in X of length at most L is $\mathrm{Vol}(M_{\widehat{\gamma}})$ bounded below by $C_X \ell_X(\gamma)$?

There are two possible notions of generic. The first possibility comes by using the ergodic properties of the geodesic flow and was introduced in [Lal14]. Using ergodic theory and the lower bound of Theorem 11 it is not hard to show that generically the volume blows up with respect to the length of the random geodesic. Another approach would be to fix a generating set G of $\pi_1(S)$ and consider random walks and see if one can obtain volume bounds in terms of the word length. The first thing to show is that asymptotically almost surely a random walk produces filling primitive geodesics.

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